JOURNAL OF ALGEBRA **207,** 43–71 (1998)
ARTICLE NO. JA987461

Linkage by Generically Gorenstein, Cohen–Macaulay Ideals

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Communicated by Melvin Hochster

Received July 29, 1996

In this paper, we study linkage by a wider class of ideals than the complete intersections. We are most interested in how the Cohen–Macaulay property behaves along this more general notion of linkage. In particular, if ideals A and B are linked by a generically Gorenstein Cohen–Macaulay ideal I, and if A is a Cohen–Macaulay ideal, we give a criterion for B to be a Cohen–Macaulay ideal. When R/B is not Cohen–Macaulay, we can give in many cases an easy description of the non-Cohen–Macaulay locus of R/B, and also a criterion for R/B to have almost maximal depth. © 1998 Academic Press

1. INTRODUCTION

In commutative algebra and algebraic geometry the notion of linkage by a complete intersection, which we will here call *classical linkage*, has long been an interesting and effective topic. Though its study apparently goes back to M. Noether in the classification of space curves, the fundamental paper [PS] of C. Peskine and L. Szpiro is widely regarded as the modern beginning of the study of classical linkage. That paper established the most important of the basic properties of linkage and, additionally, showed its usefulness in other areas, particularly algebraic geometry. Recently, there has been some interest in extending the definition of linkage to include linkage by more general ideals. In [G], E. Golod showed that the basic properties of classical linkage hold also for linkage by perfect Gorenstein ideals; in [Sc2], P. Schenzel used the theory of dualizing complexes to

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0021-8693/98 $25.00

extend the basic properties of linkage to linkage by Gorenstein ideals, and also extended to this case some important invariance results of classical linkage due to A. Rao [R]. A. Kustin and M. Miller in [KM] showed that Gorenstein linkage is very closely related to their "big-from-small" construction of Gorenstein ideals. A version of linkage by Cohen−Macaulay ideals is implicit in the work of C. Huneke [Hun1, Hun2]; in the context of linkage in a regular local ring, this might be called "linkage by k-fold hypersurface ideals."

The purpose of this paper is to study a general theory of linkage by Cohen−Macaulay ideals. Properly speaking, the linkage we consider here is linkage by generically Gorenstein, Cohen−Macaulay ideals. We are primarily interested in extending the basic results of classical linkage to this more general setting. In particular, an important theorem of classical linkage states that such linkage preserves the Cohen−Macaulay property. That is, if ideals A and B are linked by a complete intersection, and if A is a Cohen−Macaulay ideal, then B is also a Cohen−Macaulay ideal. The main results of Section 2 and 3 concern the preservation of the Cohen−Macaulay property, in its various measures, along linkage by generically Gorenstein, Cohen−Macaulay ideals. Simple examples show that the Cohen−Macaulay property itself is often not preserved; however, we are able to give relatively easy characterizations for when it is, and if it is not, we also give some results concerning preservation of depth, and we describe the non−Cohen−Macaulay locus. In Section 4, we look at a basic construction given in [PS] which, when ideals A and B are classically linked and A is a perfect ideal, produces a free resolution of R/B from a resolution of R/A. This construction is problematic for the general notion of linkage considered here; however, in some cases a naive approach yields some interesting results.

In this section, we collect together the definitions and notation to be used in this paper, and state some of the basic results of linkage theory. Most, or all, of this material is well known; sometimes, however, due to lack of a good reference, we will indicate a proof. As our primary tool is the theory of canonical modules, we will include a short discussion of this, indicating the main results we will use.

Our setting will always be in a Gorenstein local ring R. At times, we may also require R to be regular local. An ideal I of R is said to be a Cohen−Macaulay ideal, or a Gorenstein ideal, if the residue ring R/I has the corresponding property. In particular, an ideal I is generically Gorenstein provided $I_{\mathfrak{p}}$ is a Gorenstein ideal of $R_{\mathfrak{p}}$, for each minimal prime $\mathfrak{p}$ of I.

Our basic definition is:

DEFINITION 1.1. Let R be a local Gorenstein ring, and let A and B be ideals of R. Then A and B are said to be *linked* by an ideal I if

$I \subseteq A \cap B$, and if $A = (I : B)$ and $B = (I : A)$. Equivalently, A and B are linked by I if

$$A/I \cong \operatorname{Hom}_R(R/B, R/I) \quad \text{and} \quad B/I \cong \operatorname{Hom}_R(R/A, R/I).$$

We note that this is a very general definition. When I is a complete intersection, this definition coincides with the classical notion of linkage, e.g., in [PS, Sect. 2]. When I is a Gorenstein ideal, the definition corresponds to Gorenstein linkage defined in [Sc2, Def. 2.1] and [KM, Def. 1.3(2)]. In this paper, we will restrict the class of linking ideals to the generically Gorenstein, Cohen–Macaulay ideals, and, unless otherwise stated, by *linkage*, we will always mean linkage by such an ideal.

The next proposition shows how the colon ideals behave with respect to primary decompositions. It is well known.

PROPOSITION 1.2. *Let $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ be an irredundant primary decomposition of I, where $\mathfrak{q}_i$ is associated with $\mathfrak{p}_i$. Let A be another ideal. By renumbering the $\mathfrak{q}_i$, suppose that $A \not\subseteq \mathfrak{p}_i$ for $i = 1, \dots, k$, that $\mathfrak{q}_i \subset A \subseteq \mathfrak{p}_i$, for $i = k+1, \dots, l$, and that $A \subseteq \mathfrak{q}_i$, for $i = l+1, \dots, n$. Then a primary decomposition* (*not necessarily irredundant*) *for $(I : A)$ is given by*

$$(I : A) = \bigcap_{i=1}^{k} \mathfrak{q}_i \cap \bigcap_{i=k+1}^{l} (\mathfrak{q}_i : A).$$

Proof. The proof of this is easy. First note that

$$(I : A) = \left(\bigcap_i \mathfrak{q}_i : A \right) = \bigcap_i (\mathfrak{q}_i : A).$$

For those $\mathfrak{q}_i$ which contain A, $(\mathfrak{q}_i : A) = R$; if $A \not\subseteq \mathfrak{p}_i$, then $(\mathfrak{q}_i : A) = \mathfrak{p}_i$; and if $\mathfrak{q}_i \subsetneq A \subseteq \mathfrak{p}_i$, then $(\mathfrak{q}_i : A)$ is primary to $\mathfrak{p}_i$. ∎

In the context of linkage, this has the following consequence:

COROLLARY 1.3. *If A and B are ideals which are linked by I, then*

$$\operatorname{Ass}(R/A) \cup \operatorname{Ass}(R/B) = \operatorname{Ass}(R/I).$$

In particular, since in this paper, the linking ideal I is always assumed to be at least Cohen–Macaulay, it is unmixed of pure height g, say. Hence also A and B are unmixed of pure height g.

One of the primary reasons we are restricting this study to linkage by a generically Gorenstein, Cohen–Macaulay ideal is that this kind of linkage has a symmetry property. In other words, one of the relations $A = (I : B)$, $B = (I : A)$ implies the other. Although different version of this appear in [Hun1, Remark 0.2] and [Sc2, Prop. 2.2], neither statement is exactly what we need. So we state it formally here and combine the proofs from those two references. See also [PS, Prop. 2.1].

LEMMA 1.4. *Let R be a Cohen–Macaulay local ring and I a Cohen–Macaulay ideal. Suppose A is an unmixed ideal containing I, with* height A = height I. *Put $B = (I : A)$. If $R_{\mathfrak{p}}$ is Gorenstein for each minimal prime $\mathfrak{p}$ of A and $I_{\mathfrak{p}}$ is a Gorenstein ideal, then $A = (I : B)$.*

Proof. By considering the Cohen–Macaulay ring R/I in place of R, we may assume that height $A = 0$ and $B = (0 : A)$. We need to show that $A = (0 : B) = (0 : (0 : A))$, and since the forward containment always holds, we need only see that $(0 : (0 : A)) \subseteq A$. For this, it suffices to show $(0 : (0 : A_{\mathfrak{p}})) \subseteq A_{\mathfrak{p}}$ for each minimal prime $\mathfrak{p}$ over A. But $R_{\mathfrak{p}}$ is zero-dimensional Gorenstein for each such prime by assumption, and in these rings every ideal J satisfies $J = (0 : (0 : J))$ [HK, Satz 1.44]. In particular, this is true when $J = A_{\mathfrak{p}}$. ∎

It is in general an interesting question when $A = (I : B)$ implies $B = (I : A)$. As above, if I is generically Gorenstein, this statement does hold. More generally, if A is an unmixed ideal containing a Cohen–Macaulay ideal I, in some sense the number of possible B so that $A = (I : B)$ depends on the Cohen–Macaulay type of $I_{\mathfrak{p}}$, for the minimal primes $\mathfrak{p}$ not associated with A. More precisely, we have the following result.

PROPOSITION 1.5. *Let R be a Gorenstein local ring, and let I be a Cohen–Macaulay ideal of height g. Suppose A is an ideal of height g containing I, and let B_i, for $i = 1, \dots, s$, be unmixed ideals of height g containing I, so that $A = (I : B_i)$ for each i. If $\mathfrak{p} \in \operatorname{Ass}(R/I) \setminus \operatorname{Ass}(R/A)$, then $I_{\mathfrak{p}} = \bigcap_i B_{i\mathfrak{p}}$.*

Proof. Let $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k$ be an irredundant primary decomposition of I. Suppose $\mathfrak{q}_1$ is primary to $\mathfrak{p}$, where $\mathfrak{p}$ is not minimal over A. Thus, we have

$$I \subseteq \bigcap_i B_i \subseteq (I : A) = (\mathfrak{q}_1 : A) \cap \cdots \cap (\mathfrak{q}_k : A).$$

On localizing at $\mathfrak{p}$, and using that $\mathfrak{q}_1 \subseteq A$ but $A \not\subseteq \mathfrak{p}$, we obtain

$$I_{\mathfrak{p}} \subseteq \bigcap_i B_{i\mathfrak{p}} \subseteq (\mathfrak{q}_1 : A)_{\mathfrak{p}} = \mathfrak{q}_{1\mathfrak{p}} = I_{\mathfrak{p}}.$$

Hence equality holds throughout, and we obtain the result. ∎

Remark. Thus, if $r := r(R_{\mathfrak{p}}/I_{\mathfrak{p}})$ denotes the Cohen–Macaulay type of $R_{\mathfrak{p}}/I_{\mathfrak{p}}$, then if $s > r$, the set B_i of ideals is redundant at $\mathfrak{p}$. That is, there are containment relations among the $B_{i\mathfrak{p}}$. In this sense, the number of B_i possible is bounded above at $\mathfrak{p}$ by r. In particular, if I is Gorenstein at $\mathfrak{p}$, all the $B_{i\mathfrak{p}}$ are equal.

EXAMPLE. We cannot improve this in general to include associated primes of A. In $R = k[x, y]$, take $I = (x^3, x^2y, xy^2, y^3)$, $B_1 = (x^2, y)$, and $B_2 = (x, y^2)$. Then we have $(I : B_1) = (I : B_2) = (x^2, xy, y^2)$. Note that $\mathfrak{p} = (x, y) \in \operatorname{Ass}(R/A) = \operatorname{Ass}(R/I)$ and $I_{\mathfrak{p}} = (x^3, x^2y, xy^2, y^3)$, while $B_{1\mathfrak{p}} \cap B_{2\mathfrak{p}} = (x^2, xy, y^2)$.

In the last part of this paper, we shall very briefly be concerned with geometric linkage; hence, we define it here and remark that it is just linkage with an additional assumption on the associated primes.

DEFINITION 1.6. Ideals A and B of the Gorenstein local ring R are *geometrically linked* if they are each unmixed of pure height g, have no common primary components, and $I := A \cap B$ is generically Gorenstein, Cohen–Macaulay.

Note that Proposition 1.2 implies that if A and B are geometrically linked by I, then they are linked by I. Conversely, we have the following well-known result. Its proof is easy and appears in [Sc2, Lemma 2.3] for instance.

LEMMA 1.7. *If A and B are unmixed ideals, linked by the generically Gorenstein, Cohen–Macaulay ideal I, and if A and B have no common components, then A and B are geometrically linked by I, i.e., $I = A \cap B$.*

Our primary tool as we study linkage by generically Gorenstein, Cohen–Macaulay ideals will be the theory of canonical modules. At least for Cohen–Macaulay rings, this is developed in [HK]. Another excellent reference for material on canonical modules, especially for non-Cohen–Macaulay rings, is the book of Schenzel [Sc1]. We will give below the general definition of a canonical module; because we will generally work over a Gorenstein ring, we can specialize the definition, and extend it, as in [Sc1], to modules.

In the following remarks, let R be a local ring, without any *a priori* restrictions. Denote by $H^i_{\mathfrak{m}}(—)$ the local cohomology functors, and by $E(M)$ the injective hull of the module M.

Over complete local rings, the following theorem defines canonical modules:

THEOREM 1.8 [HK, Satz 5.2]. *Let R be a complete local ring*, $\dim R = n$. *Then the functor on modules defined by $M \mapsto H^n_{\mathfrak{m}}(M)$ is representable*; *that is, there exists a module K_R such that*, *for each module M*, *there is a functorial* *isomorphism*

$$H^n_{\mathfrak{m}}(M)^\vee \cong \operatorname{Hom}_R(M, K_R),$$

where, for a module N, $N^\vee$ denotes the Matlis dual $\operatorname{Hom}_R(N, E(R)/\mathfrak{m}))$ *of N.*

DEFINITION 1.9 [HK, Def. 5.6]. *A canonical module for a local ring R is a module K_R such that $K_R \otimes \hat{R} \cong K_{\hat{R}}$, where $\hat{R}$ is the completion of R, and $K_{\hat{R}}$ is the module in Theorem* 1.8.

We note that, for arbitrary local rings, a canonical module may not exist. When it does, however, it is unique [HK, Bemerkung 5.7]. For sufficiently good rings, the canonical module does exist and takes a particularly nice form. We summarize these in the following:

PROPOSITION 1.10. (1) [HK, Satz 5.9] *If R is a Gorenstein ring, then a canonical module K_R for R exists, and $K_R \cong R$. Conversely, if R is Cohen–Macaulay, and if it has a canonical module K_R with $K_R \cong R$, then R is Gorenstein.*

(2) [HK, Satz 5.12] *Suppose $S \to R$ is a local, surjective homomorphism of rings, and suppose a canonical module K_S exists for S. Then a canonical module exists for R, and $K_R \cong \operatorname{Ext}^d_S(R, K_S)$, where* $d = \dim S - \dim R$.

(3) *In particular, if R is a Gorenstein ring, and if I is an ideal of height g, then the residue class ring R/I has a canonical module, and $K_{R/I} \cong \operatorname{Ext}^g_R(R/I, R)$.*

Since we will always be working over a Gorenstein base ring, we will take (3) as our working definition for canonical modules. Following Schenzel [Sc1], we extend this definition to an arbitrary module:

DEFINITION 1.11. Let R be a local Gorenstein ring, and let M be a module; put $d = \dim R - \dim M$. Then the canonical module of M is $K_M = \operatorname{Ext}^d_R(M, R)$.

In particular, we can speak of the canonical module of a canonical module, which we will denote by $K_{K_{R/I}}$ for an ideal I of R.

An important property of canonical modules which we will require is

PROPOSITION 1.12. *Suppose M is a Cohen–Macaulay module over a Gorenstein local ring R. Then K_M is also Cohen–Macaulay.*

Proof. If $M = R/I$, this is well known; see, for example, from [HK, Satz 6.1(d)]. For arbitrary modules, it appears in the proof of [Sc1, Lemma 3.1.1(c)]. ∎

We note that even in the case $M = R/I$, if R/I is not Cohen–Macaulay, the depth of $K_{R/I}$ is quite difficult to get a handle on. See, for instance, [A], where there are constructed examples showing that the canonical module may have any possible depth. We also note that the converse of Proposition 1.12 does not hold. On the other hand, K_M always satisfies the Serre condition S_2 [Sc1, Lemma 3.1.1(c)], so for $M = R/I$, the first case where $K_{R/I}$ is not Cohen–Macaulay is for $\dim R/I = 3$. In $R = k[u, v, x, y, z]_{(u, v, x, y, z)}$, one such ideal is $I = (u^3, u^2 v, uv^2, v^3, u^2 x - uvy - v^2 z)$.

We already gave in Lemma 1.4 one reason for restricting this study to linkage by generically Gorenstein, Cohen–Macaulay ideals. Our other main reason for this restriction is that if I is such an ideal in a Gorenstein ring, the canonical module of R/I is embeddable as an ideal in R/I. Indeed, we have the following result:

LEMMA 1.13 [HK, Kor. 6.7, 6.13]. *Let R be a Cohen–Macaulay ring of dimension at least* 1, *which possesses a canonical module K_R. Then K_R is isomorphic to an ideal of R if and only if $R_{\mathfrak{p}}$ is Gorenstein, for each minimal prime $\mathfrak{p}$. Furthermore, if $K_R \cong J$ is an ideal of R, then J can be chosen to be a height* 1 *Gorenstein ideal of R.*

Thus, in particular, when I is a generically Gorenstein, Cohen–Macaulay ideal of height g in the Gorenstein ring R, then $K_{R/I}$ is isomorphic to J/I for some height $g + 1$ Gorenstein ideal J containing I.

Finally, in the proofs of many of our results, we will chase depths along exact sequences. For this, we will use the Depth Lemma, which is well known and easy to prove.

DEPTH LEMMA. *Suppose there is a short exact sequence of modules*

$$0 \to A \to B \to C \to 0$$

over a Noetherian ring R. Then one of the following holds:

$$\operatorname{depth} A \geq \operatorname{depth} B = \operatorname{depth} C,$$

$$\operatorname{depth} B \geq \operatorname{depth} A = \operatorname{depth} C + 1,$$

$$\operatorname{depth} C > \operatorname{depth} A = \operatorname{depth} C.$$

2. THE COHEN–MACAULAY PROPERTY ALONG A GENERAL LINKAGE

Our starting point in this section is the following theorem:

THEOREM 2.1. *Let R be a local Gorenstein ring. Suppose A and B are linked by an ideal I, where I is a complete intersection or a Gorenstein ideal. If A is a Cohen–Macaulay ideal, so is B.*

The case that I is a complete intersection was first shown in [PS, Prop. 1.3]. For I a perfect Gorenstein ideal, this was shown in [G]. Also, Schenzel gave a different proof in [Sc2], which works for any Gorenstein ideal. We note that easy examples show that Theorem 2.1 is false if we allow I to be a generically Gorenstein, Cohen–Macaulay ideal. The following example is perhaps the simplest; it was given in [Sc2].

EXAMPLE 2.2. Let $R = k[x, y, z, w]_{(x,y,z,w)}$ be the ring of polynomials in four variables, localized at the origin. Put $A = (y, z)$ and $B = (x, y) \cap (z, w) = (xz, yz, xw, yw)$. Then A and B are height 2 ideals linked by $I = (x, y) \cap (y, z) \cap (z, w) = (xy, yz, yw)$. Evidently, I is generically Gorenstein (in fact, generically a complete intersection). To see that it is Cohen–Macaulay, just note that it is the ideal of maximal minors of the matrix

$$\begin{pmatrix} x & y & 0 \\ w & 0 & z \end{pmatrix},$$

whence it is Cohen–Macaulay by the results of [EN]. Here, though, A is a complete intersection, hence in particular, R/A is Cohen–Macaulay, but R/B has depth 1. Indeed, it is easy to verify that $x + w$ is a maximal regular sequence on R/B.

Our purpose in this section and the next is to investigate how much of the Cohen–Macaulay property is preserved along linkage by generically Gorenstein, Cohen–Macaulay ideals. In particular, if A and B are linked by a generically Gorenstein, Cohen–Macaulay ideal I, and A is Cohen–Macaulay, we give conditions for B to be a Cohen–Macaulay ideal, and when it is not Cohen–Macaulay, we can describe its non–Cohen–Macaulay locus, and give a criterion for R/B to have high depth.

Recall that since I is generically Gorenstein, Cohen–Macaulay, Proposition 1.13 shows that there is a Gorenstein ideal J, containing I, so that J/I is a height 1 ideal of R/I and the canonical module $K_{R/I}$ of R/I is isomorphic to J/I. We note that J is highly nonunique. Also, for future reference, we note that if a prime $\mathfrak{p}$ contains I, then $I_{\mathfrak{p}}$ is a Gorenstein ideal of $R_{\mathfrak{p}}$ if and only if either J is not contained in $\mathfrak{p}$ or $J_{\mathfrak{p}} = (I_{\mathfrak{p}}, c)$, for some element $c \in R_{\mathfrak{p}}$ which is a non-zero-divisor on $R_{\mathfrak{p}}/I_{\mathfrak{p}}$. This is because in either case, $K_{R_{\mathfrak{p}}/I_{\mathfrak{p}}} = (K_{R/I})_{\mathfrak{p}} = J_{\mathfrak{p}}/I_{\mathfrak{p}} \cong R_{\mathfrak{p}}/I_{\mathfrak{p}}$, whence $R_{\mathfrak{p}}/I_{\mathfrak{p}}$ is Gorenstein, by Proposition 1.10(1).

All of our results will depend on the interaction of A and J, where $K_{R/I} \cong J/I$, for some choice of J. As such, we begin with some preliminary information on this interaction.

LEMMA 2.3. *Let R be a local ring for which the zero ideal is unmixed, A any ideal of height ≤ 1, and J any ideal of height 1. Then there exists a non-zero-divisor c of R such that A is contained in a minimal prime of cJ.*

Proof. By the height condition on A, there certainly exists a prime $\mathfrak{p}$ containing A with height $\mathfrak{p} = 1$. Thus $\mathfrak{p}$ is minimal over some non-zero-divisor c. Clearly $cJ \subseteq \mathfrak{p}$. Since J is height 1, it contains a non-zero-divisor; hence, cJ also contains a non-zero-divisor, so is of height at least 1. Since it is contained in the height 1 prime $\mathfrak{p}$, $\mathfrak{p}$ must be minimal over cJ. By construction, $\mathfrak{p}$ contains A, so we are done. ∎

COROLLARY 2.4. *Suppose $I \subseteq A$ and that I is generically Gorenstein and Cohen–Macaulay. Then $K_{R/I} \cong J/I$, where J is a Gorenstein ideal of height one more than I, and where A does not contain a non-zero-divisor for R/J.*

Proof. Since I is generically Gorenstein, Cohen–Macaulay, there exists a Gorenstein ideal J of height 1 over I such that $K_{R/I} \cong J/I$. But the ideals A/I and J/I of R/I satisfy the conditions of Lemma 2.3, so for some non-zero-divisor $\bar{c}$ of R/I, A is contained in a minimal prime for $\bar{c}J/I$. However, as R/I modules, we have $J/I \cong \bar{c}J/I$; hence, letting J' be the complete preimage in R of $\bar{c}J/I$, we see that $K_{R/I} \cong J'/I$ and A is contained in a minimal prime for J'. ∎

Note that the minimal primes of J are also minimal primes of J'. Thus, if A and B are linked by a generically Gorenstein, Cohen–Macaulay ideal I, then we can choose a J such that both A and B are contained in some minimal prime of J. In particular, $\dim R/(A + J) = d - 1$, where $d := \dim R/A$.

On the other hand, it may sometimes be possible to choose an ideal J so that height $A + J >$ height J, i.e., A contains a non-zero-divisor for R/J. Our next lemma gives a necessary and sufficient condition for this to hold, and we will see later that, often, such a choice of J may not be possible (Corollary 2.12).

LEMMA 2.5. *Let A and B be linked by the generically Gorenstein, Cohen–Macaulay ideal I with R/A Cohen–Macaulay and suppose J is such that $K_{R/I} \cong J/I$. Then A contains a non-zero-divisor for R/J if and only if $B \subseteq J$.*

Proof. First, suppose $x \in A$ is a non-zero-divisor for R/J. If $b \in B$, then $xb \in I \subseteq J$, whence $b \in J$. Thus $B \subseteq J$.

Conversely, suppose $B \subseteq J$. Then since R/A is a homomorphic image of R/I, and of the same dimension, by Proposition 1.10 we have

$$K_{R/A} \cong \mathrm{Hom}(R/A, K_{R/I}) = \mathrm{Hom}(R/I, J/I) = ((I : A) \cap J)/I$$
$$= (B \cap J)/I = B/I.$$

Next, if we apply the functor $\operatorname{Hom}(R/A, -)$ to the short exact sequence

$$0 \to J/I \xrightarrow{i} R/I \to R/J \to 0,$$

we obtain a long exact sequence

$$0 \to \operatorname{Hom}(R/A, J/I) \xrightarrow{i^*} \operatorname{Hom}(R/A, R/I) \to \operatorname{Hom}(R/A, R/J) \to 0.$$

As above $\operatorname{Hom}(R/A, J/I) = K_{R/I} \cong B/I$, and by the linkage $\operatorname{Hom}(R/A, R/I) = B/I$. It is now easily checked that the map i^* commutes with these isomorphisms, which shows that it is an isomorphism, and hence $\operatorname{Hom}(R/A, R/J) = 0$. This means that A contains a non-zero-divisor for R/J. ∎

We would like to show that, when A is a Cohen–Macaulay ideal, and J is chosen so that $\operatorname{height}(A + J) = \operatorname{height} J$, then $A + J$ is nearly a Cohen–Macaulay ideal, in the sense that $\operatorname{depth} R/(A + J) \geq \dim R/(A + J) - 1$. However, we have been unable to prove this or to find a counterexample. On the other hand, this property is definitely independent of the choice of the ideal J, as the next proposition shows. This result will be used often throughout the remainder of this chapter.

PROPOSITION 2.6. *Suppose A and B are linked by the generically Gorenstein, Cohen–Macaulay ideal I. Put $d := \dim R/I$, and write $K_{R/I} \cong J/I$. If R/A is Cohen–Macaulay, then*

(1) *if $R/(A + J)$ is Cohen–Macaulay of dimension $d - 1$, then $K_{R/B}$ is Cohen–Macaulay*;

(2) *otherwise*, $\operatorname{depth} K_{R/B} = \operatorname{depth} R/(A + J) + 2$.

Proof. First, since A and B are linked, we have

$$\begin{aligned} K_{R/B} &\cong \operatorname{Hom}(R/B, K_{R/I}) = \operatorname{Hom}(R/B, J/I) = ((I : B) \cap J)/I \\ &= (A \cap J)/I. \end{aligned}$$

Now, since $A \cap J$ has embedded components (namely, the primary components of J not containing A), it cannot be a Cohen–Macaulay ideal. Hence, from the depth lemma applied to the short exact sequence

$$0 \to (A \cap J)/I \to R/I \to R/(A \cap J) \to 0$$

we see that $\operatorname{depth}(A \cap J)/I = \operatorname{depth} R/(A \cap J) + 1$.

Next, note that there is an isomorphism $A/(A \cap J) \cong (A + J)/J$. The depth lemma applied to the sequence

$$0 \to A/(A \cap J) \to R/(A \cap J) \to R/A \to 0,$$

along with the fact that $R/(A \cap J)$ is not Cohen–Macaulay, shows that depth $R/(A \cap J) =$ depth $A/(A \cap J)$. A third application of the depth lemma on the sequence

$$0 \to (A + J)/J \to R/J \to R/(A + J) \to 0$$

shows that if depth $R/(A + J) =$ depth R/J, then depth$(A + J)/J =$ depth $R/(A + J)$; otherwise, depth$(A + J)/J =$ depth $R/(A + J) + 1$.

Finally, note that height $A + J \geq$ height J; hence if depth $R/(A + J)$ $=$ depth R/J, then $R/(A + J)$ is Cohen–Macaulay of dimension $d - 1$. Thus, we have the desired conclusion. ∎

Note that because $K_{R/B}$ does not depend on J, we have some invariance properties for the ideal $A + J$ as J runs among the ideals for which $K_{R/I} \cong J/I$. One useful such property is isolated in the following statement.

COROLLARY 2.7. *Let $d \coloneqq \dim R/I$. If $K_{R/B}$ is Cohen–Macaulay, then for every choice of J so that $K_{R/I} \cong J/I$,*

(1) *if A contains a non-zero-divisor for R/J, then* height $A + J =$ height $J + 1$, *and $R/(A + J)$ is Cohen–Macaulay*;

(2) *otherwise*, depth $R/(A + J) \geq \dim R/(A + J) - 1$.

Proof. For (1) note that certainly height $A + J \geq$ height $J + 1$. Thus, $\dim R/(A + J) \leq d - 2$, and so, from Proposition 2.6, we see that

$$d = \operatorname{depth} K_{R/B} = \operatorname{depth} R/(A + J) + 2 \leq \dim R/(A + J) + 2 \leq d.$$

Hence, depth $R/(A + J) = \dim R/(A + J) = d - 2$, showing that $R/(A + J)$ is Cohen–Macaulay, with height $A + J =$ height $J + 1$.

The second statement follows similarly: the assumption that A not contain a non-zero-divisor for R/J implies that $\dim R/(A + J) = d - 1$. If $R/(A + J)$ is not Cohen–Macaulay, then Proposition 2.6(2) shows that depth $R/(A + J) = d - 2$. In either case, then, we have depth $R/(A + J) \geq \dim R/(A + J) - 1$. ∎

Proposition 2.6 also allows us to obtain our first Cohen–Macaulay criterion for R/B, when B is linked to a Cohen–Macaulay ideal:

COROLLARY 2.8. *If* depth $R/(A + J) \geq \dim R/J - 1$, *then R/B is Cohen–Macaulay if and only if R/B satisfies the Serre condition S_2.*

Proof. Since under the hypothesis $K_{R/B}$ is Cohen–Macaulay, this follows immediately from [Sc1, Satz 3.2.3]. ∎

In some sense, Corollary 2.8 is unsatisfactory, for it characterizes the Cohen–Macaulay property for R/B in terms of R/B. Ideally, we want to use conditions on A and I alone to give a Cohen–Macaulay criterion for R/B; this is accomplished in Corollary 2.11 below. To do this, we need to extend a result of Schenzel [Sc2, Theorem 3.1] to our situation of more general linkage.

Since R is a Gorenstein local ring, it possesses a finite injective resolution, say $D^\bullet$. This complex $D^\bullet$ is a *dualizing complex* for R. That is, it is a bounded complex of injective modules, with finitely generated cohomology, such that, for every bounded complex $G^\bullet$ with finitely generated cohomology, the natural map of complexes

$$G^\bullet \to \operatorname{Hom}(\operatorname{Hom}(G^\bullet, D^\bullet), D^\bullet)$$

induces isomorphisms on cohomology (cf. [Sh] or [Ha]). For a module M the first nonvanishing cohomology of the complex $\operatorname{Hom}(M, D^\bullet)$ is $\operatorname{Ext}_R^g(M, R) = K_M$, ($g := \dim R - \dim M$), by local duality and the characterization of dimension in terms of the local cohomologies of M. Thus, we have an exact sequence of complexes in the derived category

$$0 \to K_M[-g] \to \operatorname{Hom}(M, D^\bullet) \to \mathscr{J}^\bullet(M) \to 0, \qquad (*)$$

where $\mathscr{K}^\bullet(M)$, the *truncated dualizing complex* of M, is the factor complex of the left-hand embedding. Note that from the long exact sequence on cohomology associated with $(*)$,

$$H^i(\mathscr{J}^\bullet(M)) = \begin{cases} 0, & \text{if } i \le g, \\ \operatorname{Ext}_R^i(M, R), & \text{otherwise}. \end{cases}$$

THEOREM 2.9. *Suppose A and B are linked by a generically Gorenstein, Cohen–Macaulay ideal I. Write $K_{R/I} \cong J/I$, for a Gorenstein ideal J of height* 1 *over I. Then there exists a quasi-isomorphism*

$$\mathscr{J}^\bullet(J/(A \cap J))[g] \to \operatorname{Hom}(\mathscr{J}^\bullet(R/B), D^\bullet).$$

Proof. First, recall that $K_{R/B} \cong (A \cap J)/I$, as in the proof of Proposition 2.6. Hence we have a short exact sequence

$$0 \to K_{R/B} \to J/I \to J/(A \cap J) \to 0.$$

But since R/I is Cohen–Macaulay, then in the derived category, there is an isomorphism $J/I[-g] \cong \operatorname{Hom}(R/I, D^\bullet)$. Hence the exact sequence gives rise to an exact sequence in the derived category:

$$0 \to K_{R/B}[-g] \to \operatorname{Hom}(R/I, D^\bullet) \to J/(A \cap J)[-g] \to 0.$$

Using the morphism $\mathrm{Hom}(R/B, D^\bullet) \to \mathrm{Hom}(R/I, D^\bullet)$ induced by the surjection $R/I \to R/B$ we obtain a commutative diagram of exact sequences in the derived category:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_{R/B}[-g] & \longrightarrow & \mathrm{Hom}(R/I, D^\bullet) & \longrightarrow & J/(A\cap J)[-g] & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \phi\uparrow & & \\
0 & \longrightarrow & K_{R/B}[-g] & \longrightarrow & \mathrm{Hom}(R/B, D^\bullet) & \longrightarrow & \mathscr{J}^\bullet(R/B) & \longrightarrow & 0
\end{array}$$

where $\phi^\bullet$ is induced from the first two maps. On applying the dualizing functor $\mathrm{Hom}(-, D^\bullet)$, we obtain the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}(J/(A\cap J), D^\bullet)[g] & \longrightarrow & R/I & \longrightarrow & \mathrm{Hom}(K_{R/B}, D^\bullet)[g] & \longrightarrow & 0 \\
 & & \psi^\bullet\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{Hom}(\mathscr{J}^\bullet(R/B), D^\bullet) & \longrightarrow & R/B & \longrightarrow & \mathrm{Hom}(K_{R/B}, D^\bullet)[g] & \longrightarrow & 0
\end{array} \tag{$*$}$$

where $\psi^\bullet$ is the dual of $\phi^\bullet$. We just need to show that $H^0(\mathrm{Hom}(\mathscr{J}^\bullet(R/B), D^\bullet)) = 0$ and that ψ^i induces isomorphisms on the homology level, for $i > 0$. Thus taking the long exact sequences on homology associated with the diagram $(*)$, we obtain commutative diagrams

$$\begin{array}{ccccc}
0 & \longrightarrow & \mathrm{Ext}_R^g(J/(A\cap J), R) & \longrightarrow & R/I \longrightarrow \\
 & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^0(\mathrm{Hom}(\mathscr{J}^\bullet(R/B), D^\bullet)) & \longrightarrow & R/B \longrightarrow
\end{array}$$

$$\begin{array}{ccccc}
\mathrm{Ext}_R^g(K_{R/B}, R) & \longrightarrow & \mathrm{Ext}_R^{g+1}(J/(A\cap J), R) & \longrightarrow & 0 \\
\| & & \| & & \\
\mathrm{Ext}_R^g(K_{R/B}, R) & \longrightarrow & H^1(\mathrm{Hom}(\mathscr{J}^\bullet(R/B), D^\bullet)) & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccc}
\mathrm{Ext}_R^{g+i}(K_{R/B}, R) & \xrightarrow{\sim} & \mathrm{Ext}_R^{g+i+1}(J/(A\cap J), R) \\
\downarrow & & \downarrow \\
\mathrm{Ext}_R^{g+i}(K_{R/B}, R) & \xrightarrow{\sim} & H^{i+1}(\mathrm{Hom}(\mathscr{J}^\bullet(R/B), D^\bullet))
\end{array}$$

The five-lemma applied to the first diagram shows the last map is an isomorphism, and the second diagram has the right-hand maps isomorphisms. These maps are just the maps induced on homology by ψ^i, $i > 0$. But since R/B is unmixed, it follows from [Sc1, Satz 3.2.2] that $H^0(\mathrm{Hom}(\mathscr{J}^\bullet(R/B), D^\bullet)) = 0$, and so we are finished with the proof. ∎

Remark. We recover the result of [Sc2, Theorem 3.1] when I is Gorenstein, for then $K_{R/I} \cong J/I$ is isomorphic to R/I, whence $(A+J)/J \cong R/A$. Thus, $\mathscr{J}^\bullet(R/A)[g] \cong \operatorname{Hom}(\mathscr{J}^\bullet(R/B), D^\bullet)$.

COROLLARY 2.10. *If A is Cohen–Macaulay and linked to B by the generically Gorenstein ideal I, where $K_{R/I} = J/I$, then $\operatorname{Ext}_R^{g+2}(R/(A+J), R)$ is isomorphic to the cokernel of the natural map*

$$R/B \to \operatorname{Ext}_R^g(K_{R/B}, R).$$

Proof. The proof of the above Theorem 2.9 shows that there is an exact sequence

$$0 \to R/B \to \operatorname{Ext}_R^g(K_{R/B}, R) \to \operatorname{Ext}_R^{g+1}(J/(A \cap J), R) \to 0.$$

But clearly $J/(A \cap J) \cong (A+J)/A$. Using that R/A is Cohen–Macaulay and taking the long exact sequence on Ext associated to the short exact sequence

$$0 \to (A+J)/A \to R/A \to R/(A+J) \to 0$$

we see that $\operatorname{Ext}_R^{g+1}((A+J)/A, R) \cong \operatorname{Ext}_R^{g+2}(R/(A+J), R)$, which finishes the proof. ∎

The next two corollaries establish the main Cohen–Macaulay properties of R/B. In the first, we give a strong characterization for R/B to be Cohen–Macaulay, which in contrast to our earlier characterization of Corollary 2.8 depends only on A and J. The second corollary uses this characterization to get some information on the non–Cohen–Macaulay locus of R/B.

COROLLARY 2.11. *Suppose A is Cohen–Macaulay and linked to B by the generically Gorenstein ideal I. Write $K_{R/I} \cong J/I$. Then R/B is Cohen–Macaulay if and only if $A+J$ is a Cohen–Macaulay ideal with* $\operatorname{height} A + J = \operatorname{height} J$.

Proof. Put $g = \operatorname{height} I$ and $d = \dim R/I$. Suppose that R/B is Cohen–Macaulay. Then the natural map $R/B \to K_{K_{R/B}}$ is an isomorphism [Sc2, Satz 3.2.2]. Thus from Corollary 2.10, we see that $\operatorname{Ext}_R^{g+2}(R/(A+J), R) = 0$. Next, from Proposition 2.6, we have $\operatorname{depth} R/(A+J) \geq \operatorname{depth} K_{R/B} - 1$. Since R/B is Cohen–Macaulay, so is $K_{R/B}$, necessarily of the same dimension. Hence there is only one nonvanishing Ext module for $R/(A+J)$ and this is $\operatorname{Ext}_R^{g+1}(R/(A+J), R)$. This shows $R/(A+J)$ is Cohen–Macaulay of dimension $d-1$.

Conversely, suppose $R/(A+J)$ is Cohen–Macaulay of dimension $d-1$. Then, in particular, $\operatorname{Ext}_R^{g+2}(R/(A+J), R) = 0$, showing that the natural map $R/B \to K_{K_{R/B}}$ is an isomorphism. But Proposition 2.6(1) shows that $K_{R/B}$ is Cohen–Macaulay; hence, $K_{K_{R/B}} \cong R/B$ is also Cohen–Macaulay. ∎

Remark. We recover the fact that Gorenstein linkage preserves the Cohen–Macaulay property because if I is Gorenstein, then $J = (I, c)$, for some non-zero-divisor c over R/I. Hence $A + J = (A, c)$; since c is a non-zero-divisor for R/I, it is also a non-zero-divisor for R/A. Thus, height$(A, c) =$ height $A + 1$, and $R/(A, c)$ is Cohen–Macaulay. By the corollary, this implies R/B is Cohen–Macaulay.

The next result is immediate.

COROLLARY 2.12. *If there is an ideal J with $K_{R/I} \cong J/I$ and so that A contains a non-zero-divisor for R/J, then R/B is not Cohen–Macaulay.*

For the following results, define the *non–Cohen–Macaulay locus* of a module M to be

$$\operatorname{NCM}(M) := \left\{ \mathfrak{p} \in \operatorname{Supp}(M) \colon M_{\mathfrak{p}} \text{ is not Cohen–Macaulay} \right\}.$$

COROLLARY 2.13. *Suppose* height $A + J =$ height J. *Then*

$$\operatorname{NCM}(R/B) = \operatorname{NCM}(R/(A+J)).$$

Proof. Let $\mathfrak{p} \in \operatorname{NCM}(R/(A+J))$. First, we claim $\mathfrak{p} \in \operatorname{Supp}(R/B)$. For if not, then $R_{\mathfrak{p}} = B_{\mathfrak{p}} = (I_{\mathfrak{p}} : A_{\mathfrak{p}})$, and hence $A_{\mathfrak{p}} = I_{\mathfrak{p}}$. Since $I_{\mathfrak{p}} \subseteq J_{\mathfrak{p}}$, we have $A_{\mathfrak{p}} + J_{\mathfrak{p}} = I_{\mathfrak{p}} + J_{\mathfrak{p}} = J_{\mathfrak{p}}$, and this is a Cohen–Macaulay ideal (even Gorenstein), which contradicts that $\mathfrak{p} \in \operatorname{NCM}(R/(A+J))$. Thus, in the ring $R_{\mathfrak{p}}$, we have the following situation: the proper ideal $B_{\mathfrak{p}}$ is linked to the Cohen–Macaulay ideal $A_{\mathfrak{p}}$ by the generically Gorenstein, Cohen–Macaulay ideal $I_{\mathfrak{p}}$, and $K_{R_{\mathfrak{p}}/I_{\mathfrak{p}}} = J_{\mathfrak{p}}/I_{\mathfrak{p}}$. But since $p \in \operatorname{NCM}(R/(A+J))$, $(A+J)_{\mathfrak{p}}$ is a non–Cohen–Macaulay ideal. Applying Corollary 2.11 to this situation, we see that $B_{\mathfrak{p}}$ is not a Cohen–Macaulay ideal.

Conversely, suppose $\mathfrak{p} \in \operatorname{NCM}(R/B)$, i.e, $\mathfrak{p} \in \operatorname{Supp}(R/B)$ and $R_{\mathfrak{p}}/B_{\mathfrak{p}}$ is not Cohen–Macaulay. Then we claim that $\mathfrak{p}$ must contain both A and J. If $A \not\subseteq \mathfrak{p}$, then $B_{\mathfrak{p}} = (I_{\mathfrak{p}} : A_{\mathfrak{p}}) = I_{\mathfrak{p}}$ would be Cohen–Macaulay, a contradiction. If $J \not\subseteq \mathfrak{p}$, then $I_{\mathfrak{p}}$ is a Gorenstein ideal. Thus $B_{\mathfrak{p}}$ is Gorenstein linked to the Cohen–Macaulay ideal $A_{\mathfrak{p}}$, and by Theorem 2.1, this means $B_{\mathfrak{p}}$ is a Cohen–Macaulay ideal, again a contradiction. In particular, this means that $\mathfrak{p} \in \operatorname{Supp}(R/(A+J))$. By Corollary 2.11 again, since $R_{\mathfrak{p}}/B_{\mathfrak{p}}$ is not Cohen–Macaulay, we have that $(A+J)_{\mathfrak{p}}$ is not a Cohen–Macaulay ideal, which shows that $\mathfrak{p} \in \operatorname{NCM}(R/(A+J))$. ∎

Though the proof of the previous result only covers the case that A does not contain a non-zero-divisor for R/J, a similar result holds when A does contain a non-zero-divisor for R/J. Though the basic ideas of the previous proof, with an application of Corollary 2.7, would suffice to prove this, we offer below a somewhat different proof.

PROPOSITION 2.14. *If there exists J such that* $\operatorname{height}(A+J) > \operatorname{height} J$ (*i.e., if A contains a non-zero-divisor for J*), *then $A+J$ defines the non−Cohen−Macaulay locus of R/B. That is,* $\operatorname{NCM}(R/B) = \operatorname{Supp}(R/(A+J))$.

Proof. Since A contains a non-zero-divisor for J, B must be contained in J. Thus,

$$K_{R/A} = \operatorname{Hom}_R(R/A, J/I) = ((I:A) \cap J)/I = (B \cap J)/I = B/I,$$

and, in particular, B/I is Cohen−Macaulay of dimension $d := \dim R/I$. So from the short exact sequence

$$0 \to B/I \to R/I \to R/B \to 0 \tag{1}$$

we obtain a long exact sequence on $\operatorname{Ext}_R(-, R)$:

$$0 \to K_{R/B} \to K_{R/I} \to K_{B/I} \to \operatorname{Ext}_R^{g+1}(R/B, R) \to 0. \tag{2}$$

But applying the Depth Lemma to (1), we see that $\operatorname{depth} R/B \geq d-1$. Hence, $\operatorname{NCM}(R/B) = \operatorname{Supp}(\operatorname{Ext}_R^{g+1}(R/B, R))$. Since $K_{R/B} \cong (A \cap J)/I$, and $K_{B/I} = K_{K_{R/A}} = R/A$, the sequence (2) shows that $\operatorname{Ext}_R^{g+1}(R/B, R) \cong R/(A+J)$. So $\operatorname{NCM}(R/B) = \operatorname{Supp}(R/(A+J))$. ∎

Remark. Note that the above proof shows that $R/(A+J) \cong \operatorname{Ext}_R^{g+1}(R/B, R)$. Since the module $\operatorname{Ext}_R^{g+1}(R/B, R)$ is unchanged as J is varied among the ideals so that $K_{R/I} \cong J/I$ and A contains a non-zero-divisor for R/J, then also $R/(A+J)$, and with it the ideal $A+J$, is unchanged.

Our next two results give some information on the dimension of the non−Cohen−Macaulay locus of R/B, under the additional hypothesis that $K_{R/B}$ is Cohen−Macaulay. Of course, if A contains a non-zero-divisor for R/J, then Proposition 2.14 shows, in particular, that $\dim \operatorname{NCM}(R/B) = d-2$, where $d = \dim R/B$. Thus we need only be concerned with the case that no such J exists.

THEOREM 2.15. *Suppose A and B are linked by I and that $K_{R/B}$ is Cohen−Macaulay. Choose J so that $K_{R/I} \cong J/I$ and* $\operatorname{height} A+J = \operatorname{height} J$. *Then* $\dim \operatorname{NCM}(R/B) \geq \operatorname{depth} R/B - 1$, *with equality holding if and only if the module* $\operatorname{Ext}_R^{g+2}(R/(A+J), R)$ *is Cohen−Macaulay.*

Proof. First, from Corollary 2.10 there is a short exact sequence

$$0 \to R/B \to K_{K_{R/B}} \to \operatorname{Ext}_R^{g+2}(R/(A+J), R) \to 0.$$

Moreover, the hypothesis that $K_{R/B}$ is Cohen–Macaulay implies that $K_{K_{R/B}}$ is also Cohen–Macaulay; hence, using the Depth Lemma on the exact sequence above, depth $R/B = \operatorname{depth} \operatorname{Ext}_R^{g+2}(R/(A+J), R) + 1$.

Now, we also know from Corollary 2.7 that depth $R/(A+J) \geq \dim R/(A+J) - 1$. Thus, $\operatorname{Ext}_R^{g+2}(R/(A+J), R)$ is the only non-vanishing higher Ext module; in particular, $\operatorname{NCM}(R/(A+J)) = \operatorname{Supp} \operatorname{Ext}_R^{g+2}(R/(A+J), R)$, and thus using Proposition 2.13, we have the following:

$$\begin{aligned} \dim \operatorname{NCM}(R/B) &= \dim \operatorname{NCM}(R/(A+J)) \\ &= \dim \operatorname{Ext}_R^{g+2}(R/(A+J), R) \\ &\geq \operatorname{depth} \operatorname{Ext}_R^{g+2}(R/(A+J), R) \\ &= \operatorname{depth} R/B - 1. \end{aligned}$$

The inequality is an equality if and only if $\operatorname{Ext}_R^{g+2}(R/(A+J), R)$ is Cohen–Macaulay, and this finishes the proof. ∎

EXAMPLE 2.16. We may have a strict inequality in Theorem 2.15. Let $R = k[u, v, x, y, z]$, and put

$$\begin{aligned} I = \big(&u^3v - uvxz - u^2yz, u^2v^2 - v^2xz - uvyz, u^3y - v^2z^2, \\ &u^2xy - uv^2z + vyz^2\big), \\ A = \big(&vz, u^2y, u^2v\big). \end{aligned}$$

It is easily checked that I is generically Gorenstein, Cohen–Macaulay, and A is Cohen–Macaulay, both of height 2. On the other hand $B := (I : A)$ has depth 1, but $\operatorname{NCM}(R/B) = \mathscr{V}(a_3a_4)$, where $a_i = \operatorname{ann} \operatorname{Ext}_R^i(R/B, R)$ [Sc1, Satz 2.4.6], and this has dimension 1. Hence, $\dim \operatorname{NCM}(R/B) = 1 > 0 = \operatorname{depth} R/B - 1$.

THEOREM 2.17. *Let R be an n-dimensional Gorenstein ring and A and B height g ideals linked by a generically Gorenstein, Cohen–Macaulay ideal I, with $K_{R/I} \cong J/I$, and R/A Cohen–Macaulay. Suppose $K_{R/B}$ is Cohen–Macaulay and also that $(A+J)/J$ satisfies the Serre condition S_r, but not S_{r+1}. Then* $\dim \operatorname{NCM}(R/B) = n - g - r - 1$.

Proof. As in the proof of Theorem 2.15, we have

$$\dim \mathrm{NCM}(R/B) = \dim \mathrm{Ext}_R^{g+2}(R/(A+J), R).$$

Now, from the short exact sequence

$$0 \to (A+J)/A \to R/A \to R/(A+J) \to 0,$$

and using that R/A is Cohen–Macaulay, we see that $\mathrm{Ext}_R^{g+i}((A+J)/A, R) \cong \mathrm{Ext}_R^{g+i}(R/(A+J), R)$, for $i \geq 1$. Also, since $K_{R/B}$ is Cohen–Macaulay, by Corollary 2.7, depth $R/(A+J) \geq \dim R/(A+J) - 1$; hence, all the modules $\mathrm{Ext}_R^i(R/(A+J), R)$ vanish, for $i > g+2$, and so the corresponding Ext modules for $(A+J)/A$ also vanish. Furthermore, the module $(A+J)/A$ is equidimensional, since its associated primes are associated primes of the Cohen–Macaulay ideal A. Thus we can apply [Sc1, Lemma 3.2.1]. This says that $\dim \mathrm{Ext}_R^{g+1}((A+J)/A, R) = n - g - 1 - r$, where r is the largest integer so that $(A+J)/A$ satisfies S_r. ∎

3. ON A LOWER BOUND FOR THE DEPTH OF LINKED IDEAL

Suppose that A and B are linked by a generically Gorenstein, Cohen–Macaulay ideal I and that R/A is Cohen–Macaulay. In the previous section, we concentrated, for the most part, on when R/B was Cohen–Macaulay, i.e., when R/B had maximal depth. In this section, we are concerned with when R/B has high depth. By this, we mean that depth $R/B \geq \dim R/B - 1$; thus, R/B is nearly Cohen–Macaulay. We will say in this case that R/B (or B) "satisfies the depth inequality." One of the condition which imply this will lead naturally to further consideration, and we are able to show that a result of Peskine and Szpiro on the sum of A and B continues to be valid in many cases. On the other hand, we give an example where it fails.

We have already seen one case when the above depth inequality holds, namely, when $K_{R/I} \cong J/I$, where J is a Gorenstein ideal of height 1 over I and A contains a non-zero-divisor for J. For in this case, we must have $B \subseteq J$, and hence

$$K_{R/A} = \mathrm{Hom}_R(R/A, J/I) = ((I : A) \cap J)/I = B/I$$

is Cohen–Macaulay. Thus, the Depth Lemma applied to the exact sequence

$$0 \to B/I \to R/I \to R/B \to 0$$

shows that depth $R/B \geq \dim R/B - 1$.

The next example shows that this is not a necessary condition:

EXAMPLE 3.1. Let $R = k[u, v, x, y, z, w]$ be the polynomial ring in six variables. Let I be the ideal generated by the 2×2 minors of the matrix

$$\begin{pmatrix} u - v & w^2 & y & wx \\ x - w & 0 & x + z & yz \end{pmatrix}.$$

Thus, I has height 3 and is Cohen–Macaulay. Also, $K_{R/I} \cong J/I$, where $J = (u - v, x - w)^3 + I$, so R/I is generically Gorenstein. If we let B be the unmixed part of the ideal $(x, w, z)^2 + I$ and put $A = (I : B)$, then it is easily checked, by MACAULAY for example, that A is a Cohen–Macaulay ideal, B satisfies the depth inequality, but $K_{R/A} \not\cong B/I$. Thus, A does not contain a non-zero-divisor for J, for any J with $J/I \cong K_{R/I}$.

We note also that the depth inequality is not always satisfied. The following example was communicated to me by C. Walter. The methods of his paper [W], especially those of Remark 1.1, allow many other similar examples to be easily computed.

EXAMPLE 3.2. Let $R = k[u, v, w, y, z]$ be the polynomial ring in five indeterminates. Let B be the ideal defining the Veronesean surface in $\mathbb{P}^4$. That is, B is the kernel of the obvious map

$$k[u, v, x, y, z] \to k\left[\alpha^2, \beta^2, \gamma^2, \beta(\alpha - \gamma), \gamma(\alpha - \beta)\right].$$

Put $A = (x - z, vz - yz)$ and $I = A \cap B$. Then it is easy to check that I is a generically Gorenstein, Cohen–Macaulay ideal. Thus A and B are linked by I, R/A is Cohen–Macaulay, but R/B has depth 1. In particular, B does not satisfy the depth inequality.

Recall from Corollary 2.11 that R/B is Cohen–Macaulay if and only if $R/(A + J)$ is Cohen–Macaulay of dimension $d - 1$, where $d = \dim R/A$. Our characterization of when R/B satisfies the depth inequality is similar, except that it only involves the unmixed part of $A + J$. Unfortunately, we have to make the additional assumption that $K_{R/B}$ is Cohen–Macaulay; as stated previously, we believe that this is already implied by the linkage, but have been unable to prove it.

Notation. For an ideal G, we denote by G' the unmixed part of G. That is, G' is the intersection of the primary components of G of highest dimension. Since we are in a Gorenstein local ring, note that $G' = \operatorname{ann} \operatorname{Ext}_R^{n-g}(R/G, R)$, where $g = \text{height } G$.

We first need a lemma, which is well known; cf. [Mat, Exercise 6.4]:

LEMMA 3.3. *Let R be a Cohen–Macaulay ring, and let A and B be ideals of the same height with $B \subseteq A$. Denote by A', resp. B', the intersection of the* *minimal primary components of A, resp. B. Then $B' = A'$ if and only if $B_{\mathfrak{p}} = A_{\mathfrak{p}}$ for each minimal prime $\mathfrak{p}$ of B.*

Proof. The forward direction is trivial, so assume that $B_{\mathfrak{p}} = A_{\mathfrak{p}}$ for each minimal prime of B. Let $B' = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ and $A' = Q_1 \cap \cdots \cap Q_m$ be primary decompositions. Then it will suffice to show that $n = m$, and that $\mathfrak{q}_i = Q_i$, perhaps after renumbering. First, since A and B have the same height, if P is a minimal prime of A, it is also minimal over B. Thus, $m \leq n$. On the other hand, if $\mathfrak{p}$ is minimal over B, then since $A_{\mathfrak{p}} = B_{\mathfrak{p}}$ is a proper ideal of $R_{\mathfrak{p}}$, we see that $\mathfrak{p}$ is minimal over A, as well, and that the primary components of A and B corresponding to $\mathfrak{p}$ are equal. Thus $m = n$ and $\mathfrak{q}_i = Q_i$ for each i, up to renumbering. ∎

PROPOSITION 3.4. *Let R be a local Cohen–Macaulay ring of dimension n. Suppose the ideals A and B have height $n-1$, and are linked by the generically Gorenstein, Cohen–Macaulay ideal I, where R/I has canonical module isomorphic to J/I, for a Gorenstein ideal J of height 1 over I. Then $A + J$ and $B + J$ are linked by J.*

Proof. By working modulo I, we reduce to the case that R is a one-dimensional Cohen–Macaulay ring, A and B are height 0 ideals such that $A = (0 : B)$ and $B = (0 : A)$, and J is isomorphic to the canonical module of R. We need to show that $\mathrm{Hom}_R(R/B, R/J) \cong (A + J)/J$ and $\mathrm{Hom}_R(R/A, R/J) \cong (B + J)/J$. It will clearly suffice to prove just one of these, as the proof of the other will be similar, and, in any case, follows from Lemma 1.4.

Now, there is a natural short exact sequence

$$0 \to J \to R \to R/J \to 0,$$

which give rise to the long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_R(R/B, J) &\to \mathrm{Hom}_R(R/B, R) \to \\ &\mathrm{Hom}_R(R/B, R/J) \to \mathrm{Ext}^1_R(R/B, J) \to \cdots . \end{aligned} \tag{1}$$

By the linkage of A and B, we have natural isomorphisms

$$\mathrm{Hom}_R(R/B, J) \cong A \cap J \quad \text{and} \quad \mathrm{Hom}_R(R/B, R) \cong A.$$

Also, since R/B is unmixed in the one-dimensional CM ring R, it is Cohen–Macaulay; in particular, the local cohomology $H^0_{\mathfrak{m}}(R/B) = 0$. But R is Cohen–Macaulay with canonical module J; thus, by the local duality theorem and the homological characterization of depth, $\mathrm{Ext}^1_R(R/B, J) \cong H^0_{\mathfrak{m}}(R/B)^{\vee} = 0$, where $M^{\vee}$ is the Matlis dual of M. Thus, the sequence

(1) gives rise to a short exact sequence

$$0 \to A \cap J \to A \to \operatorname{Hom}_R(R/B, R/J) \to 0,$$

where the first map is the inclusion. That is, $\operatorname{Hom}_R(R/B, R/J) \cong A/(A \cap J) \cong (A + J)/J$, as required. ∎

THEOREM 3.5. *Suppose A and B are linked by a generically Gorenstein, Cohen–Macaulay ideal I in the local Gorenstein ring R. Write $K_{R/I} \cong J/I$ for a Gorenstein ideal J of height 1 over I, and suppose neither A nor B contains a non-zero-divisor for R/J. Then $(A + J)'$ and $(B + J)'$ are linked by J.*

Proof. First, we have $(J:(B + J)') = (J : B + J)$, by considering a primary decomposition of $B + J$, and using that J is unmixed. Next, since A and B are linked, we clearly have $A + J \subseteq (J : B + J)$. Thus, by Lemma 3.3, we have only to show that $(A + J)_{\mathfrak{p}} = (J : B + J)_{\mathfrak{p}}$, for each minimal prime $\mathfrak{p}$ of $A + J$.

Now, such a minimal prime $\mathfrak{p}$ will always contain A and J. If $B \not\subseteq \mathfrak{p}$, then choosing an element $x \in B \setminus \mathfrak{p}$, for each $a \in A$, we have $ax \in I \subseteq J$, showing that A is contained in the $\mathfrak{p}$-primary component of J. Thus, on localizing at $\mathfrak{p}$, we have $A_{\mathfrak{p}} \subseteq J_{\mathfrak{p}}$, and hence $(A + J)_{\mathfrak{p}} = J_{\mathfrak{p}} = (J_{\mathfrak{p}} : R_{\mathfrak{p}}) = (J_{\mathfrak{p}} : B_{\mathfrak{p}} + J_{\mathfrak{p}})$, so the equality holds in this case.

Thus, we are reduced to the case that $\mathfrak{p}$ contains B. By localizing at $\mathfrak{p}$ we are in the situation described by Proposition 3.4, and the proof of the theorem is finished. ∎

With these preliminaries, we are able to prove the main result of this section. It establishes a necessary and sufficient condition in terms of A and J for R/B to have high depth. As we have already noted, this condition is quite similar in spirit to the one of Corollary 2.11.

PROPOSITION 3.6. *Suppose the Cohen–Macaulay ideal A is linked to B by the generically Gorenstein, Cohen–Macaulay ideal I, and write $K_{R/I} \cong J/I$, where neither A nor B contains a non-zero-divisor for R/J. Suppose that $K_{R/B}$ is Cohen–Macaulay. Then $(A + J)'$ is a Cohen–Macaulay ideal if and only if R/B satisfies the depth inequality.*

Proof. Note that Corollary 2.11 shows that R/B is Cohen–Macaulay if and only if $R/(A + J)$ is Cohen–Macaulay. Thus, for the rest of the proof, we may assume that neither of these properties holds.

Now, since $K_{R/B}$ is Cohen–Macaulay, then from Corollary 2.7(2) and the depth lemma applied to the sequence

$$0 \to (A + J)/J \to R/J \to R/(A + J) \to 0, \qquad (*)$$

we see that $(A + J)/J$ is Cohen–Macaulay of the same dimension as R/J. In particular, on applying the functor $\operatorname{Hom}(-, R)$ to $(*)$, we obtain a long exact sequence

$$\begin{aligned}0 \to \operatorname{Ext}_R^{g+1}(R/(A+J), R) &\to \operatorname{Ext}_R^{g+1}(R/J, R) \to \\ \operatorname{Ext}_R^{g+1}((A+J)/J, R) &\to \operatorname{Ext}_R^{g+2}(R/(A+J), R) \to 0. \qquad (1)\end{aligned}$$

Now, the cokernel of the first map in this sequence is isomorphic to $R/(B+J)'$. This follows from the commutative diagram

$$\begin{tikzcd}
\operatorname{Ext}_R^{g+1}(R/(A+J), R) \arrow[r] \arrow[d] & \operatorname{Ext}_R^{g+1}(R/(A+J)', R) \arrow[r] \arrow[d] & \operatorname{Hom}(R/(A+J)', R/J) \arrow[r] \arrow[d] & (B+J)'/J \arrow[d] \\
\operatorname{Ext}_R^{g+1}(R/J, R) \arrow[r] & \operatorname{Ext}_R^{g+1}(R/J, R) \arrow[r] & \operatorname{Hom}(R/, R/J) \arrow[r] & R/J
\end{tikzcd} \qquad (2)$$

The first square of (2) follows from applying $\operatorname{Hom}(-, R)$ to the diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & (A+J)/J \arrow[r] \arrow[d] & R/J \arrow[r] \arrow[d] & R/(A+J) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (A+J)'/J \arrow[r] & R/J \arrow[r] & R/(A+J)' \arrow[r] & 0
\end{tikzcd}$$

The second square of (2) follows from [G, Lemma 2]. Finally, the last square follows from Theorem 3.5.

As the vertical map on the left of (2) is induced from the natural surjection of R/J onto $R/(A+J)'$, and all the horizontal maps are natural isomorphisms, the cokernel of this map is isomorphic to the cokernel of the natural injection of $(B+J)'/J$ into R/J, which is what we wanted to show.

Thus, the exact sequence (1) gives rise to an exact sequence

$$\begin{aligned}0 \to R/(B+J)' \to \operatorname{Ext}_R^{g+1}&((A+J)/J, R) \to \\ &\operatorname{Ext}_R^{g+2}(R/(A+J), R) \to 0. \qquad (3)\end{aligned}$$

Now, the Depth Lemma applied to the exact sequence of Corollary 2.10 shows that depth $R/B = \operatorname{depth} \operatorname{Ext}_R^{g+2}(R/(A+J), R) + 1$, since we have assumed that R/B is not Cohen−Macaulay, and that $K_{R/B}$ is Cohen−Macaulay. Also, note that, since $(A+J)/J$ is Cohen−Macaulay, so too is $\operatorname{Ext}_R^{g+1}((A+J)/J, R)$.

Thus, assume that $R/(A+J)'$ is Cohen–Macaulay. Since $(A+J)'$ is linked to $(B+J)'$ by the Gorenstein ideal J, then $R/(B+J)'$ is also Cohen–Macaulay. Hence, the previous remark together with the Depth Lemma applied to (3) shows that depth $R/B =$ depth $\operatorname{Ext}_R^{g+2}(R/(A+J), R) + 1 =$ depth $R/(B+J)' = d-1$, so R/B satisfies the depth inequality. Conversely, if R/B satisfies the depth inequality, this forces depth $\operatorname{Ext}_R^{g+2}(R/(A+J), R) = d-2$, and hence $R/(B+J)'$is Cohen–Macaulay, from the Depth Lemma applied to (3), again. But now the linkage shows that $R/(A+J)'$ is also Cohen–Macaulay. ∎

The next proposition gives a different criterion for the depth inequality in terms of the sum of A and B, when A and B are geometrically linked:

PROPOSITION 3.7. *Suppose A and B are ideals of height g of a Gorenstein ring R, with A a Cohen–Macaulay ideal geometrically linked to B by a generically Gorenstein ideal I. Then B satisfies the depth inequality if and only if $A+B$ is a Cohen–Macaulay ideal of height $g+1$.*

Proof. First, application of the $\operatorname{Hom}(k, -)$ functor to the short exact sequence

$$0 \to R/B \to R/A \oplus R/B \to R/A \to 0,$$

shows that $\operatorname{Ext}_R^i(k), R/B) \cong \operatorname{Ext}_R^i(k, R/A \oplus R/B)$ for all $i < \dim R/A = d$.

Since A and B are geometrically linked by I, there is an exact sequence

$$0 \to R/I \to R/A \oplus R/B \to R/(A+B) \to 0.$$

Again, applying the functor $\operatorname{Hom}(k, -)$, we obtain

$$\operatorname{Ext}_R^j(k, R/(A+B)) \cong \operatorname{Ext}_R^j(k, R/A \oplus R/B),$$

for $j < d-1$. Thus we see that

$$\operatorname{Ext}_R^i(k, R/(A+B)) \cong \operatorname{Ext}_R^i(k, R/B), \qquad \text{for all } i < d-1.$$

Now note that, since A and B are geometrically linked, they have no common primary components, and hence $\dim R/(A+B) \leq d-1$. If $R/(A+B)$ is Cohen–Macaulay of dimension $d-1$, then the above isomorphism shows that depth $R/B \geq d-1 = \dim R/B - 1$. Conversely, if this inequality holds, the isomorphism shows depth $R/(A+B) \geq d-1 \geq \dim R/(A+B)$; thus equality holds and $R/(A+B)$ is Cohen–Macaulay of dimension $d-1$. ∎

Motivation for the above characterization is provided by [PS, Remarque 1.4], where it is shown that, when A and B are height g ideals geometrically linked by a complete intersection (or, more generally, a Gorenstein ideal) I, and if A, and hence also B, is a Cohen–Macaulay ideal, then in fact $A + B$ is a Gorenstein ideal of height $g + 1$. The next example shows that this fails, in general, for linkage by generically Gorenstein, Cohen–Macaulay ideals.

EXAMPLE 3.8. Let $R = k[x, y, z, w]_{(x,y,z,w)}$ be the polynomial ring in four variables, localized at the origin. Let $I = (w^2, x^2) \cap (z, y) \cap (w^2, y^2)$. Then I is a height 2, generically Gorenstein ideal. To see that I is Cohen–Macaulay, note that it is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} z & y & 0 \\ x^2y & 0 & w^2 \end{pmatrix}.$$

Let $A = (w^2, y^2)$ and $B = (w^2, x^2) \cap (z, y)$. Thus A and B are geometrically linked by I, and A is Cohen–Macaulay. But it is easily checked, by MACAULAY, for example, that $A + B$ is a height 3 Cohen–Macaulay ideal which is not Gorenstein.

Note, however, that $K_{R/A} \cong R/A \ncong B/I$. The next result shows that this is a consequence of $A + B$ not being Gorenstein.

PROPOSITION 3.9. *Suppose A and B are height g ideals of the Gorenstein ring R, with A Cohen–Macaulay and geometrically linked by a generically Gorenstein, Cohen–Macaulay ideal I to B. If $K_{R/A} \cong B/I$, then $A + B$ is a Gorenstein ideal of height $g + 1$.*

Proof. We have

$$K_{R/A} \cong B/I = B(A \cap B) \cong (A + B)/A \subseteq R/A.$$

By Proposition 1.12, since R/A is Cohen–Macaualy, this shows that $(A + B)/A$ is a height 1 Gorenstein ideal of R/A. Since $A \subseteq A + B$, this is equivalent to saying that $A + B$ is a height $g + 1$ Gorenstein ideal of R, as required. ∎

Remark. We recover the result of [PS, Remarque 1.4], for when A and B are linked by a Gorenstein ideal I, then, automatically, $K_{R/A} \cong B/I$. For a comprehensive study of this situation, see B. Ulrich's paper [U].

4. ON THE GENERATORS AND FREE RESOLUTIONS OF LINKED IDEALS

The results in this section arose in an attempt to generalize to the notion of linkage being considered here the following result of Peskine and Szpiro:

THEOREM 4.1. *Let R be a Gorenstein local ring, and let A and B be ideals linked by a complete intersection ideal I. Suppose A is perfect; that is, R/A is Cohen–Macaulay of finite projective dimension. Let $\mathbf{F}$ be a minimal free resolution of R/A, let $\mathbf{K}$ be the Koszul complex resolving R/I, and let α: $\mathbf{K} \to \mathbf{F}$ be a comparison map induced by the inclusion of I into A. Let $\mathbf{C}$ be the mapping cylinder of the dual map α^*: $\mathbf{F}^* \to \mathbf{K}^*$. Then $\mathbf{C}$ is a free resolution of R/B.*

It is easy to see that if we replace I by a Gorenstein ideal of finite projective dimension, and, corresondingly, $\mathbf{K}$ by a minimal free resolution of R/I, then the conclusion of Theorem 4.1 continues to hold. Essential for this is the functorial isomprhism

$$\operatorname{Ext}_R^g(R/A, R) \cong \operatorname{Hom}_R(R/A, R/I),$$

which is part of [G, Lemma 2].

Our two main results of this section are as follows. First, we wish to see what happens if we take the naive approach and just work through this mapping cylinder construction, when A and B are linked by a generically Gorenstein, Cohen–Macaulay ideal I of finite projective dimension and A is perfect. We obtain a free resolution of a quotient of canonical modules, and this quotient has the unmixed part of its first Fitting invariant equal to B. That is, its presentation matrix has maximal minors which generate an ideal whose unmixed part is B. Moreover, such a presentation matrix is easy to come by; it arises from resolutions of R/A and R/I, and the comparison map between these. Our second aim in this section is to find an alternative method of constructing a resolution of R/B. Although we cannot do this in general, we can give a construction for one easy case.

We first need to recall the notion of a mapping cylinder of a map of complexes. Suppose $(\mathbf{C}, d^C)$ and $(\mathbf{D}, d^D)$ are complexes, and let α: $\mathbf{C} \to \mathbf{D}$ be a morphism of complexes. Then the mapping cylinder $\mathscr{C} := \mathscr{C}(\alpha)$ of α is the complex with component modules $\mathscr{C}_n = \mathbf{C}_{n-1} \oplus \mathbf{D}_n$ and differentials d_n: $\mathscr{C}_n \to \mathscr{C}_{n-1}$ defined by

$$d_n(x, y) = \left(d_{n-1}^C(x), \alpha_{n-1}(x) + (-1)^n d_n^D(y)\right).$$

There is an obvious short exact sequence of complexes

$$0 \to \mathbf{D} \to \mathscr{C} \to \mathbf{C}[-1] \to 0$$

and taking the long exact sequence on homology we obtain:

LEMMA 4.2. *With the notation as above, if $\mathbf{C}$ and $\mathbf{D}$ are acyclic, then $H_i(\mathscr{C}) = 0$ for $i \geq 2$. Furthermore, if the morphism α induces an injection $H_0(\mathbf{C}) \to H_0(\mathbf{D})$, then $H_1(\mathscr{C}) = 0$, i.e., $\mathscr{C}$ is acyclic.*

For future reference, recall that if M is a finitely generated module over a Noetherian ring R, and if $\mathfrak{p} \in \operatorname{Spec} R$ is a prime ideal, then there is a functorial isomorphism

$$\operatorname{Ext}^i_R(M, N)_{\mathfrak{p}} \cong \operatorname{Ext}^i_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}).$$

We state the next lemma in somewhat more generality than we have used so far in this paper. Specifically, we suppose that A and B are ideals linked by the Cohen–Macaulay ideal I; we do not assume I to be generically Gorenstein.

LEMMA 4.3. *Let A, B, and I be as noted above. Let i: $A/I \to R/I$ be the canonical inclusion, and let $M = \operatorname{im} i^*$, where i^*: $\operatorname{Hom}(R/A, K_{R/I}) \to \operatorname{Hom}(R/I, K_{R/I})$. Then* $\operatorname{ann} M = B$.

Proof. The map i fits into an exact sequence

$$0 \to A/I \xrightarrow{i} R/I \xrightarrow{\pi} R/A \to 0.$$

Applying the functor $\operatorname{Hom}(-, K_{R/I})$, we obtain a long exact sequence

$$0 \to \operatorname{Hom}(R/A, K_{R/I}) \xrightarrow{\pi^*} \operatorname{Hom}(R/I, K_{R/I}) \xrightarrow{i^*} \operatorname{Hom}(A/I, K_{R/I}) \to \cdots .$$

Since clearly $B \subseteq \operatorname{Hom}(A/I, K_{R/I})$, also $B \subseteq \operatorname{ann} M$. For the opposite inclusion, suppose $b \in \operatorname{ann} M$. Let $a \in A$. For each $\phi \in \operatorname{Hom}(R/I, K_{R/I})$, we see that $i^*(b\phi) = bi^*(\phi) = 0$, so $b\phi = \pi^*(\psi)$ for some ψ in $\operatorname{Hom}(R/A, K_{R/I})$. Thus $ab\phi = a\pi^*(\psi) = \pi^*(a\psi) = 0$, since $a \in A = \operatorname{ann} \operatorname{Hom}(R/A, K_{R/I})$. Since ϕ was arbitrary, $ab \in \operatorname{ann} \operatorname{Hom}(R/I, K_{R/I}) = I$, and since $a \in A$ was arbitrary, $b \in (I : A) = B$. ∎

From the elementary properties of the Fitting invariants, the next corollary is immediate.

COROLLARY 4.4. *Let $\mathscr{F} := \mathscr{F}_0(M)$ denote the first Fitting invariant of M. Then* $\operatorname{rad} \mathscr{F} = \operatorname{rad} B$.

This corollary essentially says that B and $\mathscr{F}$ have the same minimal primes. When we require I to be generically Gorenstein, then they also have the same minimal primary components:

PROPOSITION 4.5. *Suppose A and B are linked by the generically Gorenstein, Cohen–Macaulay ideal I. With the notation as above, $\mathscr{F}' = B$.*

Proof. By Lemma 4.3 and the elementary properties of Fitting invariants, we have $\mathscr{F} \subseteq B$. Hence, by Lemma 3.3, we need only show that, for each minimal prime $\mathfrak{p}$ of $\mathscr{F}$, $\mathscr{F}_{\mathfrak{p}} = B_{\mathfrak{p}}$. Note that, by Corollary 4.4, such a prime $\mathfrak{p}$ is also minimal over B, hence also minimal over I.

Now, if we localize the exact sequence

$$0 \to \operatorname{Hom}\big(R/A, K_{R/I}\big) \xrightarrow{\pi_{\mathfrak{p}}^*} \operatorname{Hom}\big(R/I, K_{R/I}\big) \to M \to 0,$$

and use the functorial isomorphism above, together with the functorial identifications $\operatorname{Hom}(R/A, K_{R/I}) \cong K_{R/A}$ and $\operatorname{Hom}(R/I, K_{R/I}) \cong K_{R/I}$, we obtain the short exact sequence

$$0 \to K_{R_{\mathfrak{p}}/A_{\mathfrak{p}}} \xrightarrow{\pi_{\mathfrak{p}}^*} K_{R_{\mathfrak{p}}/I_{\mathfrak{p}}} \to M_{\mathfrak{p}} \to 0.$$

But I is generically Gorenstein, so $I_{\mathfrak{p}}$ is a Gorenstein ideal. Hence $K_{R_{\mathfrak{p}}/I_{\mathfrak{p}}} \cong R_p/I_p$, and $K_{R_{\mathfrak{p}}/A_{\mathfrak{p}}} \cong B_{\mathfrak{p}}/I_{\mathfrak{p}}$, and $\pi_{\mathfrak{p}}^*$ is the inclusion of $B_{\mathfrak{p}}/I_{\mathfrak{p}}$ into R_p/I_p. Hence, $M_{\mathfrak{p}} \cong R_{\mathfrak{p}}/B_{\mathfrak{p}}$, and this has Fitting invariant $\mathscr{F}_{\mathfrak{p}} = \mathscr{F}(M_{\mathfrak{p}}) = B_{\mathfrak{p}}$, which finishes the proof. ∎

The advantage of this description of B is that the Fitting invariant can be computed quite easily, at least when both I and A are perfect ideals. Indeed, let **F** and **G** be minimal finite free resolutions of R/I and R/A, respectively. Then as in [S, p. 57] or [G, Lemma 1], the dual complexes $\mathbf{F}^*$ and $\mathbf{G}^*$ are free resolutions of $K_{R/I}$ and $K_{R/A}$, respectively. Moreover, if $\alpha\colon \mathbf{F} \to \mathbf{G}$ is the comparison map covering the surjection $R/I \to R/A$, then α^* induces the injection $\pi^*\colon K_{R/A} \to K_{R/I}$ which appears in the proof of Proposition 4.5. By Lemma 4.2,then, the mapping cylinder of α^* is a free resolution of M. In particular, the first map in this resolution is a presentation matrix for M, and the Fitting invariant $\mathscr{F}$ is the ideal generated by the maximal minors of this matrix.

As an immediate corollary, when the Fitting invariant $\mathscr{F}$ is unmixed, we have an upper bound for the number of generators of B. For this, recall that, for a Cohen–Macaulay ideal H, $r(R/H) = \mu(K_{R/H})$, where $r(R/H)$ denotes the Cohen–Macaulay type of R/H [HK, Kor. 6.11].

PROPOSITION 4.6. *With the notation as above, suppose $\mathscr{F}$ is unmixed. Let $t := \beta_g = \mu(K_{R/I}) = r(R/I)$ and β_{g-1} be the last two nonzero Betti numbers of R/I, and let $s = r(R/A) = \mu(K_{R/A})$. Then $\mu(B) \le \binom{\beta_{g-1}+s}{t}$.*

Proof. The presentation matrix for M obtained as the previous paragraph has size $(\beta_{g-1} + s) \times t$. Since $\mathscr{F}$ is unmixed, $\mathscr{F} = B$, and this is generated by the maximal minors of the presentation matrix of M. There are exactly $\binom{\beta_{g-1}+s}{t}$ of these. ∎

Our next objective is to construct, in certain cases, a resolution of R/B in terms of resolutions of R/I and R/A. We need to make the following assumption: write $K_{R/I} \cong J/I$, and assume that $B \subseteq J$. This is equivalent, by Lemma 2.5, to saying that A contains a non-zero-divisor for R/J. Note that in this case, by the remarks preceding Example 3.1, B satisfies the depth inequality of Section 2; that is, depth $R/B \geq \dim R/B - 1$. Thus, if height $B = g$, the best resolution we can hope for will have length $g + 1$.

Now, let $\mathbf{F} \to R/I \to 0$ be a minimal free resolution of R/I. Since R/I is Cohen–Macaulay, the dual complex $\mathbf{F}^*$ is a resolution of $K_{R/I} \cong J/I$. Thus, the mapping cylinder construction applied to the diagram

$$\begin{tikzcd} \mathbf{F}^* \arrow[r] \arrow[d, "\alpha"] & J/I \arrow[r] \arrow[d] & 0 \\ \mathbf{F} \arrow[r] & R/I \arrow[r] & 0 \end{tikzcd}$$

where α is a comparison map induced by the inclusion of J/I into R/I, produces a resolution of R/J. Note that this resolution has the "right" length: height $J =$ height $I + 1$, and the resolution produced in the above construction has height $I + 1$ terms. On the other hand, if α is not a minimal lifting of the embedding $J/I \to R/I$, then the resolution will not be a minimal resolution of R/J. Also, note that in this case, the mapping cylinder construction is a kind of "symmetrization" process; that is, if $\mathbf{C}$ is the mapping cylinder of α, then $\mathbf{C} \cong \mathbf{C}^*$.

Next, let $\mathbf{G} \to R/A \to 0$ be a minimal free resolution of R/A. Again, since R/A is Cohen–Macaulay, the dual complex $\mathbf{G}^*$ is a free resolution of $K_{R/A}$. However, $K_{R/A} \cong \operatorname{Hom}_R(R/A, K_{R/I}) = ((I : A) \cap J)/I = (B \cap J)/I = B/I$, the last equality following from our assumption that $B \subseteq J$. Also, from the short exact sequence

$$0 \to A/I \to R/I \to R/A \to 0$$

and the fact that R/A is Cohen–Macaulay, by dualizing, we obtain another short exact sequence

$$0 \to K_{R/A} \to K_{R/I} \to K_{A/I} \to 0. \tag{$*$}$$

Again, the mapping cylinder construction now produces a free resolution of $K_{A/I}$ from the two complexes

$$\mathbf{G}^* \to K_{R/A} \to 0,$$

$$\mathbf{F}^* \to K_{R/I} \to 0.$$

But using the isomorphisms $K_{R/I} \cong J/I$ and $K_{R/A} \cong B/I$, the sequence $(*)$ shows that $K_{A/I} \cong J/B$.

Hence, we have constructed resolutions of J/B and R/J. Finally, the horseshoe construction applied to the short exact sequence

$$0 \to J/B \to R/B \to R/J \to 0$$

produces a resolution of R/B. We note that, under our assumption that $B \subseteq J$, we know that depth $R/B \geq \dim R/B - 1$ and that the resolution of R/B produced has exactly height $B + 1$ terms. So, in general, it is the shortest possible. On the other hand, it is definitely not a minimal resolution, as R/B is cyclic, but the rightmost free module has rank equal to $\mu(J/B) + \mu(R/J) > 1$.

REFERENCES

[A] Y. Aoyama, On the depth and the projective dimension of the canonical module, *Japan J. Math.* **6** (1980), 61–66.

[EN] J. A. Eagon and D. G. Northcott, Ideals defined by matrices and a certain complex associated with them, *Proc. Roy. Soc. Ser. A* **269** (1962), 188–204.

[G] E. S. Golod, A note on perfect ideals, *in* ‘‘Algebra’’ (A. I. Kostrikin, Ed.), pp. 37–79, Moscow State Univ. Publishing House; 1980. (Translated by L. L. Avramov.)

[Ha] R. Hartshorne, ‘‘Residues and Duality,’’ Lecture Notes in Mathematics, Vol. 20, Springer-Verlag, Berlin/New York, 1966.

[Hun1] C. Huneke, Linkage and the Koszul homology of ideals, *Amer. J. Math.* **104** (1982), 1043–1062.

[Hun2] C. Huneke, Strongly Cohen–Macaulay schemes and residual intersections, *Trans. Amer. Math. Soc.* **277** (1983), 739–763.

[HK] J. Herzog and E. Kunz, ‘‘Der Kanonische Modul Eines Cohen–Macaulay Rings,’’ Lecture Notes in Mathematics, Vol. 238, Springer-Verlag, Berlin/New York, 1971.

[KM] A. Kustin and M. Miller, Deformation and linkage of Gorenstein algebras, *Trans. Amer. Math. Soc.* **284** (1984), 501–533.

[Mat] H. Matsumura, ‘‘Commutative Ring Theory,’’ Cambridge Univ. Press, Cambridge, UK, 1989.

[PS] C. Peskine and L. Szpiro, Liaison des variétées algébriques, *Invent. Math.* **26** (1974), 271–302.

[R] A. P. Rao, Liaison des variétés algébriques, *Invent. Math.* **26** (1974), 271–302.

[R] A. P. Rao, Liaison among curves in $\mathbb{P}^3$, *Invent. Math.* **50** (1979), 205–217.

[Sc1] P. Schenzel, ‘‘Dualisierende Komplexe in der Lokalen Algebra and Buchsbaum-Ringe,’’ Lecture Notes in Mathematics, Vol. 907, Springer-Verlag, Berlin/New York, 1982.

[Sc2] P. Schenzel, Notes on liaison and duality, *J. Math. Kyoto Univ.* **22** (1982/83), 185–198.

[Sh] R. Sharp, Dualizing complexes for commutative Noetherian rings, *Math. Proc. Cambridge Philos. Soc.* **78** (1975), 369–386.

[S] R. Stanley, ‘‘Combinatorics and Commutative Algebra,’’ Birkhauser, Boston/Basel, 1983.

[U] B. Ulrich, Sums of linked ideals, *Trans. Amer. Math. Soc.* **318** (1990), 1–42.

[W] C. Walter, Cohen–Macaulay liaison, preprint.